Lattices and perfect form theory

Mathieu Dutour Sikirić

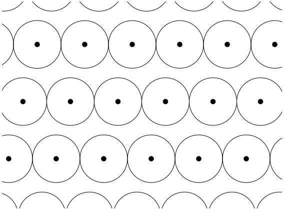
Institute Rudjer Bošković, Croatia and
Universität Rostock

February 27, 2014

I. Lattices and Gram matrices

Lattice packings

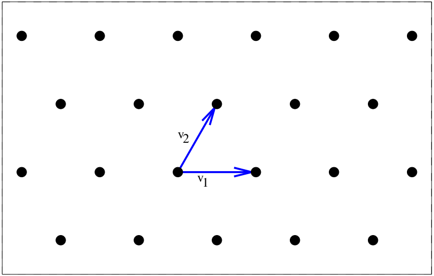
- A lattice $L \subset \mathbb{R}^n$ is a set of the form $L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$ with $(v1, \ldots, v_n)$ independent.
- A packing is a family of balls $B_n(x_i, r)$, $i \in I$ of the same radius r and center x_i such that their interiors are disjoint.



- If L is a lattice, the lattice packing is the packing defined by taking the maximal value of $\alpha > 0$ such that $L + B_n(0, \alpha)$ is a packing.
- The maximum α is called $\lambda(L)$ and the determinant of $(v_1, \ldots, v_n)$ is det L.

Gram matrix and lattices

- Denote by S^n the vector space of real symmetric $n \times n$ matrices, $S^n_{>0}$ the convex cone of real symmetric positive definite $n \times n$ matrices and $S^n_{\geq 0}$ the convex cone of real symmetric positive semidefinite $n \times n$ matrices.
- Take a basis $(v_1, \ldots, v_n)$ of a lattice L and associate to it the Gram matrix $G_{\mathbf{v}} = (\langle v_i, v_j \rangle)_{1 \leq i,j \leq n} \in S^n_{>0}$.
- Example: take the hexagonal lattice generated by $v_1 = (1, 0)$ and $v_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$



$$G_{\mathbf{v}} = \frac{1}{2} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Isometric lattices

- Take a basis $(v_1, \ldots, v_n)$ of a lattice L with $v_i = (v_{i,1}, \ldots, v_{i,n}) \in \mathbb{R}^n$ and write the matrix

$$V = \begin{pmatrix} v_{1,1} & \cdots & v_{n,1} \\ \vdots & \ddots & \vdots \\ v_{1,n} & \cdots & v_{n,n} \end{pmatrix}$$

  and $G_{\mathbf{v}} = V^T \, V$.
  The matrix $G_{\mathbf{v}}$ is defined by $\frac{n(n+1)}{2}$ variables as opposed to n^2 for the basis V.
- If $M \in S^n_{>0}$, then there exists V such that $M = V^T \, V$ (Gram Schmidt orthonormalization)
- If $M = V_1^T \, V_1 = V_2^T \, V_2$, then $V_1 = OV_2$ with $O^T \, O = I_n$ (i.e. O corresponds to an isometry of $\mathbb{R}^n$).
- Also if L is a lattice of $\mathbb{R}^n$ with basis $\mathbf{v}$ and u an isometry of $\mathbb{R}^n$, then $G_{\mathbf{v}} = G_{u(\mathbf{v})}$.

Arithmetic minimum

- The arithmetic minimum of $A \in S^n_{>0}$ is

$$\min(A) = \min_{x \in \mathbb{Z}^n - \{0\}} x^T A x$$

- The minimal vector set of $A \in S^n_{>0}$ is

$$\mathrm{Min}(A) = \left\{ x \in \mathbb{Z}^n \mid x^T A x = \min(A) \right\}$$

- Both $\min(A)$ and $\mathrm{Min}(A)$ can be computed using some programs (for example SV by Vallentin)
- The matrix $A_{hex} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has

$$\mathrm{Min}(A_{hex}) = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}.$$

Re-expression of previous definitions

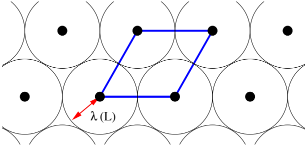
- Take a lattice $L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$. If $x \in L$,

$$x = x_1 v_1 + \cdots + x_n v_n \quad \text{with} \quad x_i \in \mathbb{Z}$$

we associate to it the column vector $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

- We get $||x||^2 = X^T G_{\mathbf{v}} X$ and

$$\det L = \sqrt{\det G_{\mathbf{v}}} \quad \text{and} \quad \lambda(L) = \frac{1}{2}\sqrt{\min(G_{\mathbf{v}})}$$



- For $A_{hex} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $\det A_{hex} = 3$ and $\min(A_{hex}) = 2$

Changing basis

- If $\mathbf{v}$ and $\mathbf{v}'$ are two basis of a lattice L then $V' = VP$ with $P \in \mathrm{GL}_n(\mathbb{Z})$. This implies

$$G_{\mathbf{v}'} = V'^T V' = (VP)^T VP = P^T \{V^T V\} P = P^T G_{\mathbf{v}} P$$

- If $A, B \in S^n_{>0}$, they are called arithmetically equivalent if there is at least one $P \in \mathrm{GL}_n(\mathbb{Z})$ such that

$$A = P^T B P$$

- Lattices up to isometric equivalence correspond to $S^n_{>0}$ up to arithmetic equivalence.
- In practice, Plesken/Souvignier wrote a program ISOM for testing arithmetic equivalence and a program AUTO for computing automorphism group of lattices.
All such programs take Gram matrices as input.

Dual lattices

- For a lattice L the dual lattice is

$$L^* = \{x \in \mathbb{R}^n \text{ s.t. } \langle x, y\rangle \in \mathbb{Z} \text{ for all } y \in L\}$$

- If $L = P\mathbb{Z}^n$ then we can take $L^* = (P^{-1})^T\mathbb{Z}^n$ and we get

$$G(L^*) = (G(L))^{-1}$$

- A lattice L is integral if $\langle x, y\rangle \in \mathbb{Z}$ for all $x, y \in \mathbb{Z}$.
- This is equivalent to say $L \subset L^*$
- A lattice is self-dual if $L = L^*$.
- A lattice is self-dual if and only if its Gram matrix is integral and of determinant 1.

Root lattices

- A root lattice is a lattice generated by a root system
- They are integral, $\|x\|^2 \in 2\mathbb{Z}$ and $\mathrm{Min}(L)$ is the root system
- Most classical example is

$$\mathrm{A}_n = \left\{ x \in \mathbb{Z}^{n+1} \text{ s.t. } \sum_{i=1}^{n+1} x_i = 0 \right\}$$

  Possible basis: $v_i = e_{i+1} - e_i$ for $1 \leq i \leq n$
- They have a strict ADE classification:

Name	Min	$\|Min\|$	det	$\|\mathrm{Aut}\|$
A_n	$e_i - e_j$	$2n(n+1)$	$n+1$	$2(n+1)!$
D_n	$\pm e_i \pm e_j$	$4n(n-1)$	4	$2^n n!$
E_6	complex	72	3	103680
E_7	complex	126	2	2903040
E_8	complex	240	1	696729600

Self-dual even lattice

- A lattice is even if for all $x \in L$, $\langle x, x \rangle \in 2\mathbb{Z}$.
- The Theta function of a self-dual even lattice of dimension n is

$$\Theta(L, q) = \sum_{x \in L} q^{\langle x, x \rangle}$$

  and it is a modular form for $\mathrm{SL}_2(\mathbb{Z})$ of weight $n/2$.
- This implies that they exist only for dimension n divisible by 8.

Dimension	lattices
8	E_8
16	$\mathrm{E}_8 \oplus \mathrm{E}_8$ and D_{16}^{+}
24	Leech lattice and 24 Niemeier lattices
32	at least 40 million lattices

- The key to above enumeration and estimates are the Siegel Mass formula and Kneser's algorithm
  - M. Kneser, *Quadratische Formen*, Springer Verlag.

The Leech lattice

- Every non-zero vector v has $\|v\|^2 \geq 4$ and $\det Leech = 1$.
- It is the best lattice packing in dimension 24. Density is

$$\frac{\pi^{12}}{12!} \simeq 0.001930...$$

- There are 196280 shortest vectors (maximal number in dimension 24)
- The covering radius is $\sqrt{2}$ and covering density is

$$\frac{\pi^{12}}{12!}\left(\sqrt{2}\right)^{24} \simeq 7.903536...$$

  It is conjectured to give the best covering in dimension 24.
- Its automorphism group quotiented by $\pm Id_{24}$ is the sporadic simple group Co_0 and it contains many sporadic simple groups as subgroups.
- It is also related to some Lorentzian lattices.

II. Computational techniques

Polytopes, definition

- A polytope $P \subset \mathbb{R}^n$ is defined alternatively as:
  - The convex hull of a finite number of points $v^1, \dots, v^m$:

$$P = \{v \in \mathbb{R}^n \mid v = \sum_i \lambda_i v^i \text{ with } \lambda_i \geq 0 \text{ and } \sum \lambda_i = 1\}$$

  - The following set of solutions:

$$P = \{x \in \mathbb{R}^n \mid f_j(x) \geq b_j \text{ with } f_j \text{ linear}\}$$

    with the condition that P is bounded.
- The cube is defined alternatively as
  - The convex hull of the 2^n vertices

$$\{(x_1, \dots, x_n) \text{ with } x_i = \pm 1\}$$

  - The set of points $x \in \mathbb{R}^n$ satisfying to

$$x_i \leq 1 \text{ and } x_i \geq -1$$

Facets and vertices

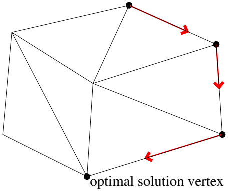
- A vertex of a polytope P is a point $v \in P$, which cannot be expressed as $v = \lambda v^1 + (1-\lambda)v^2$ with $0 < \lambda < 1$ and $v^1 \neq v^2 \in P$.
- A polytope is the convex hull of its vertices and this is the minimal set defining it.
- A facet of a polytope is an inequality $f(x) - b \geq 0$, which cannot be expressed as $f(x) - b = \lambda(f_1(x) - b_1) + (1-\lambda)(f_2(x) - b_2)$ with $f_i(x) - b_i \geq 0$ on P.
- A polytope is defined by its facet inequalities. and this is the minimal set of linear inequalities defining it.
- The dual-description problem is the problem of passing from one description to another.
- There are several programs CDD, LRS for computing dual-description computations.
- In case of large problems, we can use the symmetries for faster computation.

Linear programs

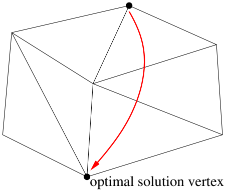
- A linear program is the problem of maximizing a linear function $f(x)$ over a set $\mathcal{P}$ defined by linear inequalities.

$$\mathcal{P} = \{x \in \mathbb{R}^d \quad \text{such that} \quad f_i(x) \geq b_i\}$$

  with f_i linear and $b_i \in \mathbb{R}$.
- The solution of linear programs is attained at vertices of $\mathcal{P}$.
- There are two classes of solution methods:



Simplex method



Interior point method

- Simplex methods use exact arithmetic but have bad theoretical complexity
- Interior point methods have good theoretical complexity but only gives an approximate vertex.

III. Perfect forms and domains

Perfect forms

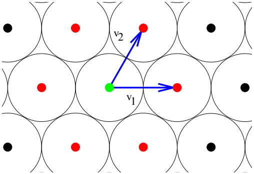
- A form A is extreme if it is a local maximum of the packing density.
- A matrix $A \in S^n_{>0}$ is perfect (Korkine & Zolotarev) if the equation

  $$B \in S^n \text{ and } x^T B x = \min(A) \text{ for all } x \in \mathrm{Min}(A)$$

  implies $B = A$.
- Theorem: (Korkine & Zolotarev) If a form is extreme then it is perfect.
- Up to a scalar multiple, perfect forms are rational.
- All root lattices are perfect, many other families are known.

A perfect form

- $A_{hex} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ corresponds to the lattice:



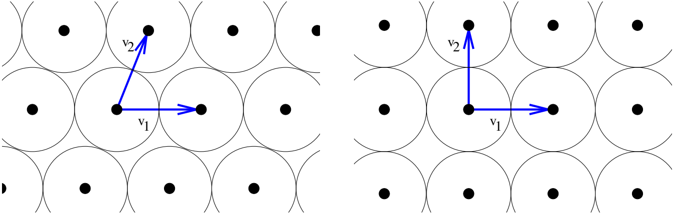
- If $B = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ satisfies to $x^T Bx = \min(A_{hex})$ for $x \in \mathrm{Min}(A_{hex}) = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}$, then:

$$a = 2, \quad c = 2 \text{ and } a - 2b + c = 2$$

which implies $B = A_{hex}$. A_{hex} is perfect.

A non-perfect form

- $A_{sqr} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has $\mathrm{Min}(A_{sqr}) = \{\pm(0,1), \pm(1,0)\}$.
- See below lattices L_B, L_{sqr} associated to matrices $B, A_{sqr} \in S^2_{>0}$ with $\mathrm{Min}(B) = \mathrm{Min}(A_{sqr})$:



Perfect domains and arithmetic closure

- If $v \in \mathbb{Z}^n$ then the corresponding rank 1 form is $p(v) = vv^T$.
- If A is a perfect form, its perfect domain is

$$\mathrm{Dom}(A) = \sum_{v \in \mathrm{Min}(A)} \mathbb{R}_+ p(v)$$

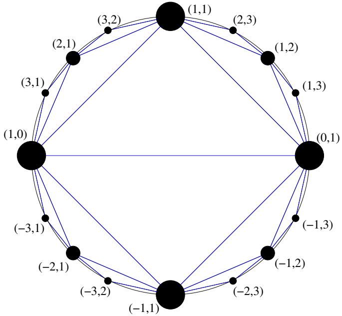
- If A has m shortest vectors then $\mathrm{Dom}(A)$ has $\frac{m}{2}$ extreme rays.
- So actually, the perfect domains realize a tessellation not of $S^n_{>0}$, nor $S^n_{\geq 0}$ but of the rational closure $S^n_{rat,\geq 0}$.
- The rational closure $S^n_{rat,\geq 0}$ has a number of descriptions:
  - $S^n_{rat,\geq 0} = \sum_{v \in \mathbb{Z}^n} \mathbb{R}_+ p(v)$
  - If $A \in S^n_{\geq 0}$ then $A \in S^n_{rat,\geq 0}$ if and only if *Ker A* is defined by rational equations.
- So, actually, the stabilizers of some faces of the polyhedral complex are infinite.

Finiteness

- Theorem:(Voronoi) Up to arithmetic equivalence there is only finitely many perfect forms.
- The group $\mathrm{GL}_n(\mathbb{Z})$ acts on $S^n_{>0}$:

$$Q \mapsto P^t Q P$$

  and we have $\mathrm{Min}(P^t Q P) = P^{-1}\,\mathrm{Min}(Q)$
- $\mathrm{Dom}(P^T Q P) = c(P)^T\,\mathrm{Dom}(Q) c(P)$ with $c(P) = \left(P^{-1}\right)^T$
- For $n = 2$, we get the classical picture:



Known results on lattice packing density maximization

dim.	Nr. of perfect forms	Best lattice packing
2	1 (Lagrange)	A_2
3	1 (Gauss)	A_3
4	2 (Korkine & Zolotarev)	D_4
5	3 (Korkine & Zolotarev)	D_5
6	7 (Barnes)	E_6 (Blichfeldt & Watson)
7	33 (Jaquet)	E_7 (Blichfeldt & Watson)
8	10916 (DSV)	E_8 (Blichfeldt & Watson)
9	$\geq$500000	Λ_9?
24	?	Leech (Cohn & Kumar)

- The enumeration of perfect forms is done with the Voronoi algorithm.
- The number of orbits of faces of the perfect domain tessellation is much higher but finite (Known for $n \leq 7$)
- Blichfeldt used Korkine-Zolotarev reduction theory.
- Cohn & Kumar used Fourier analysis and Linear programming.

Some algorithms

- Pb 1: Suppose we have a configuration of vector $\mathcal{V}$. Does there exist a matrix $A \in S^n_{>0}$ such that $\mathrm{Min}(A) = \mathcal{V}$?
- Consider the linear program

$$\begin{array}{rl} \text{minimize} & \lambda \\ \text{with} & \lambda = A[v] \text{ for } v \in \mathcal{V} \\ & A[v] \geq 1 \text{ for } v \in \mathbb{Z}^n - \{0\} - \mathcal{V} \end{array}$$

  The value λ_{opt} determines the answer.
- In practice one replaces $\mathbb{Z}^n$ by a finite set and iteratively increases it until a conclusion is reached.
- Pb 2: How given a matrix $A \in S^n_{>0}$ find B perfect with $A \in \mathrm{Dom}(B)$?
- The method is to start from a perfect matrix B and test if A belongs to $\mathrm{Dom}(B)$. If not there exist a facet F of $\mathrm{Dom}(B)$ such that A is on the other side (found by LP).
  We flip over it. Eventually, one finds the right perfect form.

IV. Ryshkov polyhedron and the Voronoi algorithm

The Ryshkov polyhedron

- The Ryshkov polyhedron R_n is defined as

$$R_n = \left\{ A \in S^n \text{ s.t. } x^T A x \geq 1 \text{ for all } x \in \mathbb{Z}^n - \{0\} \right\}$$

- The cone is invariant under the action of $\mathrm{GL}_n(\mathbb{Z})$.
- The cone is locally polyhedral, i.e. for a given $A \in R_n$

$$\left\{ x \in \mathbb{Z}^n \text{ s.t. } x^T A x = 1 \right\}$$

  is finite
- Vertices of R_n correspond to perfect forms.
- For a form $A \in R_n$ we define the local cone

$$Loc(A) = \left\{ Q \in S^n \text{ s.t. } x^T Q x \geq 0 \text{ if } x^T A x = 1 \right\}$$

The Voronoi algorithm

- Find a perfect form (say A_n), insert it to the list $\mathcal{L}$ as undone.
- Iterate
  - For every undone perfect form A in $\mathcal{L}$, compute the local cone $Loc(A)$ and then its extreme rays.
  - For every extreme ray r of $Loc(A)$ realize the flipping, i.e. compute the adjacent perfect form $A' = A + \alpha r$.
  - If A' is not equivalent to a form in $\mathcal{L}$, then we insert it into $\mathcal{L}$ as undone.
- Finish when all perfect forms have been treated.

The sub-algorithms are:

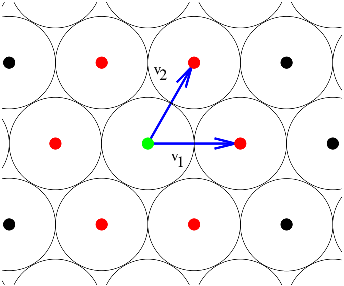
- Find the extreme rays of the local cone $Loc(A)$ (use CDD or LRS or any other program)
- For any extreme ray r of $Loc(A)$ find the adjacent perfect form A' in the Ryshkov polyhedron R_n
- Test equivalence of perfect forms using ISOM

Flipping on an edge I

$$\mathrm{Min}(A_{hex}) = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}$$

with

$$A_{hex} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

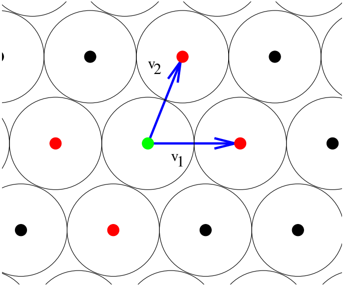


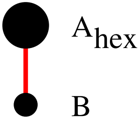
Flipping on an edge II

$$\mathrm{Min}(B) = \{\pm(1,0), \pm(0,1)\}$$

with

$$B = \begin{pmatrix} 1 & 1/4 \\ 1/4 & 1 \end{pmatrix} = A_{hex} + D/4$$



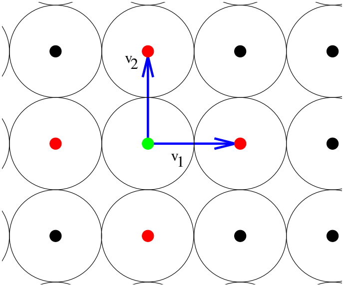


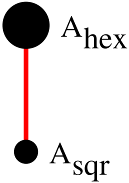
Flipping on an edge III

$$\mathrm{Min}(A_{sqr}) = \{\pm(1,0), \pm(0,1)\}$$

with

$$A_{sqr} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A_{hex} + D/2$$



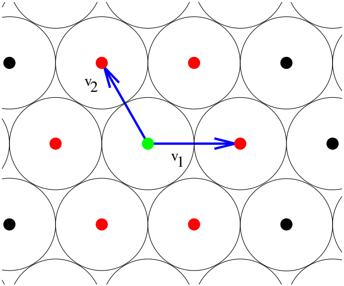


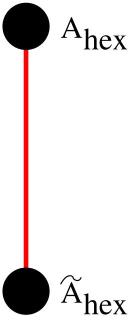
Flipping on an edge IV

$$\mathrm{Min}(\tilde{A}_{hex}) = \{\pm(1,0), \pm(0,1), \pm(1,1)\}$$

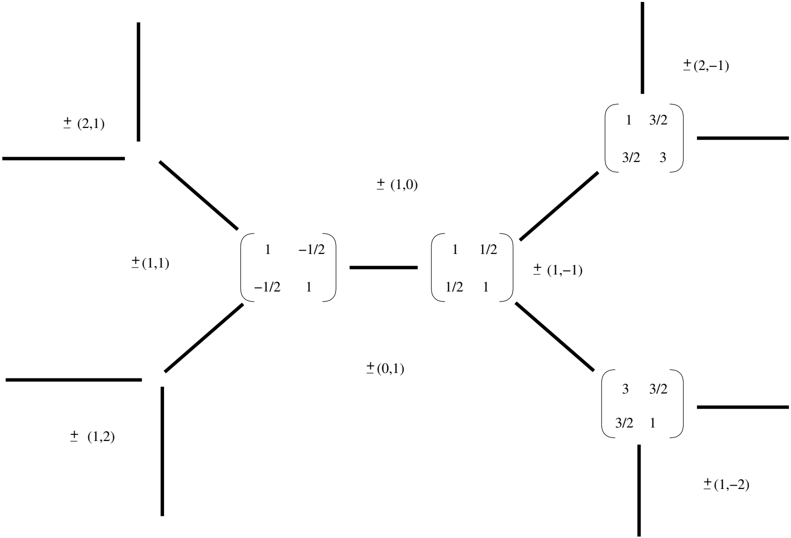
with

$$\tilde{A}_{hex} = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix} = A_{hex} + D$$





The Ryshkov polyhedron R_2



Well rounded forms and retract

- A form Q is said to be well rounded if it admits vectors v_1, $\dots$, v_n such that
  - $(v_1, \dots, v_n)$ form a $\mathbb{R}$-basis of $\mathbb{R}^n$ (not necessarily a $\mathbb{Z}$-basis)
  - $v_1, \dots, v_n$ are shortest vectors of Q.
- Well rounded forms correspond to bounded faces of R_n.
- Every form can be continuously deformed to a well rounded form and this defines a retracting homotopy of R_n onto a polyhedral complex WR_n of dimension $\frac{n(n-1)}{2}$.
- Every face of WR_n has finite stabilizer.
- Actually, in term of dimension, we cannot do better:
  - A. Pettet and J. Souto, *Minimality of the well rounded retract*, Geometry and Topology, **12** (2008), 1543-1556.
- We also cannot reduce ourselves to lattices whose shortest vectors define a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

Topological applications

- The fact that we have finite stabilizers for all faces means that we can compute rational homology/cohomology of $\mathrm{GL}_n(\mathbb{Z})$ efficiently.
- This has been done for $n \leq 7$
  - P. Elbaz-Vincent, H. Gangl, C. Soulé, *Perfect forms, K-theory and the cohomology of modular groups*, Adv. Math 245 (2013) 587–624.
- As an application, we can compute $K_n(\mathbb{Z})$ for $n \leq 8$.
- By using perfect domains, we can compute the action of Hecke operators on the cohomology.
- This has been done for $n \leq 4$:
  - P.E. Gunnells, *Computing Hecke Eigenvalues Below the Cohomological Dimension*, Experimental Mathematics 9-3 (2000) 351–367.
- The above can, in principle, be extended to the case of $\mathrm{GL}_n(R)$ with R a ring of algebraic integers.

References

On lattice theory:

- J.H. Conway and N.J.A. Sloane, *Sphere Packings, Lattices and Groups* third edition, Springer–Verlag, 1998.

On perfect forms:

- G. Voronoi, *Nouvelles applications des paramètres continues à la théorie des formes quadratiques 1: Sur quelques propriétés des formes quadratiques positives parfaites*, J. Reine Angew. Math **133** (1908) 97–178.
- A. Schürmann, *Computational geometry of positive definite quadratic forms*, University Lecture Notes, AMS.
- J. Martinet, *Perfect lattices in Euclidean spaces*, Springer, 2003.
- S.S. Ryshkov, E.P. Baranovski, *Classical methods in the theory of lattice packings*, Russian Math. Surveys **34** (1979) 1–68, translation of Uspekhi Mat. Nauk **34** (1979) 3–63.

V. Tessellations

Linear Reduction theories for S^n

Some $\mathrm{GL}_n(\mathbb{Z})$ invariant tessellations of $S^n_{rat,\geq 0}$:

- The perfect form theory (Voronoi I) for lattice packings (full face lattice known for $n \leq 7$, perfect domains known for $n \leq 8$)
- The central cone compactification (Igusa & Namikawa) (Known for $n \leq 6$)
- The L-type reduction theory (Voronoi II) for Delaunay tessellations (Known for $n \leq 5$)
- The C-type reduction theory (Ryshkov & Baranovski) for edges of Delaunay tessellations (Known for $n \leq 5$)
- The Minkowski reduction theory (Minkowski) it uses the successive minima of a lattice to reduce it (Known for $n \leq 7$) not face-to-face
- Venkov's reduction theory also known as Igusa's fundamental cone (finiteness proved by Crisalli)

Toroidal compactifications of $\mathcal{A}_g$

- A polyhedral $\mathrm{GL}_n(\mathbb{Z})$-tessellation of $S^n_{rat,\geq 0}$ is admissible if it is a face-to-face tessellation and has finite number of orbits.
- Admissible $\mathrm{GL}_n(\mathbb{Z})$ invariant tessellations of $S^n_{rat,\geq 0}$ give rise to toroidal compactifications of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties.
- For the perfect form tessellation $\mathcal{A}_g^{Perf}$ is a canonical model in the sense of the minimal model program if $g \geq 12$:
  - N. Shepherd-Barron, *Perfect forms and the moduli space of abelian varieties*, Invent. Math. 163-1 (2006) 25–45
- For Voronoi II tessellation $\mathcal{A}_g^{Vor}$ has its boundary corresponding to semi-abelic varieties:
  - V. Alexeev, *Complete moduli in the presence of semiabelian group action*, Ann. of Math. 155-3 (2002) 611–708
- Properties of the compactification being $\mathbb{Q}$-Gorenstein, having canonical singularities, terminal singularities can be read off from properties of the tessellation.

Geometry of tessellation and compactifications

- Thm: (Namikawa) For a given admissible tessellation $\mathcal{F}$ the corresponding tessellation is smooth if and only if
  - All cones are simplicial
  - For all cones, the set of generators of extreme rays can be extended to a basis of $\mathrm{Sym}^2(\mathbb{Z})$.
- For $\mathcal{A}_g^{Perf}$ we prove
  - Every cone of dimension at most 9 in the perfect cone decomposition is basic. In particular the stack $\mathcal{A}_g^{Perf}$ is smooth for $g \leq 3$ and the codimension of both the singular and the non-simplicial substack of $\mathcal{A}_g^{Perf}$ is 10 if $g \geq 4$.
  - Every cone of dimension 10 is simplicial with the only exception the cone of the root lattice D_4.
- For $\mathcal{A}_g^{Vor}$ we prove
  - For $g \leq 4$ every cone in the second Voronoi compactification is basic.
  - For $g \geq 5$ there are non-simplicial cones in dimension 3, in particular $\mathcal{A}_g^{Vor}$ is singular in dimension 3.

Self-dual cones

- For an open cone C in $\mathbb{R}^n$ the dual cone is

$$C^* = \{x \in \mathbb{R}^n \text{ s.t. } \langle x, y \rangle > 0 \text{ for } y \in C\}$$

- Such cones are classified by Euclidean Jordan algebras and the classification gives:
  - S^n: The cone of positive definite real quadratic forms
  - H^n: The cone of positive definite Hermitian quadratic forms
  - Q^n: The cone of positive definite quaternionic quadratic forms
  - The cone of 3×3 positive definite octonion matrices.
  - The hyperbolic cone H_n

$$H_n = \left\{(x_1, \dots, x_n) \text{ s.t. } x_1 > 0 \text{ and } x_1^2 - x_2^2 - \cdots - x_n^2 > 0\right\}$$

- References

  - A. Ash, D. Mumford, M. Rapoport, Y. Tai *Smooth compactifications of locally symmetric varieties*, Cambridge University Press
  - M. Koecher, *Beiträge zu einer Reduktionstheorie in Positivtätsbereichan I/II*, Math. Annalen 141, 384–432, 144, 175–182

T-space theory

- A T-space $\mathcal{F}$ is a vector space in S^n with $\mathcal{F}_{>0} = \mathcal{F} \cap S^n_{>0}$ being non-empty.
- All above reduction theories apply to that case.
- But some dead ends exist to the polyhedral tessellations.
- Relevant group is $\mathrm{Aut}(\mathcal{F}) = \{g \in \mathrm{GL}_n(\mathbb{Z}) \text{ s.t. } g\mathcal{F}g^T = \mathcal{F}\}$.
- For a finite group $G \subset \mathrm{GL}_n(\mathbb{Z})$ of space

$$\mathcal{F}(G) = \left\{ A \in S^n \text{ s.t. } gAg^T = A \text{ for } g \in G \right\}$$

  we have $\mathrm{Aut}(\mathcal{F}(G)) = Norm(G, \mathrm{GL}_n(\mathbb{Z}))$ (Zassenhaus) and a finite number of $\mathcal{F}$-perfect forms.
- There exist some T-spaces having a rational basis and an infinity of perfect forms.
- Another finiteness case is for spaces obtained from $\mathrm{GL}_n(R)$ with R number ring.

Non-polyhedral reduction theories

- Some works with non-polyhedral, but still manifold domains:
  - R. MacPherson and M. McConnel, *Explicit reduction theory for Siegel modular threefolds*, Invent. Math. **111** (1993) 575–625.
  - D. Yasaki, *An explicit spine for the Picard modular group over the Gaussian integers*, Journal of Number Theory, **128** (2008) 207–234.
- Other works in complex hyperbolic space using Poincaré polyhedron theorem:
  - M. Deraux, *Deforming the* $\mathbb{R}$*-fuchsian* $(4,4,4)$*-lattice group into a lattice*.
  - E. Falbel and P.-V. Koseleff, *Flexibility of ideal triangle groups in complex hyperbolic geometry*, Topology **39** (2000) 1209–1223.
- Other works for non-manifold setting would be:
  - T. Brady, *The integral cohomology of* $Out_+(F_3)$, Journal of Pure and Applied Algebra **87** (1993) 123–167.
  - K.N. Moss, *Cohomology of* $\mathrm{SL}(n, \mathbb{Z}[1/p])$, Duke Mathematical Journa **47-4** (1980) 803–818.

VI. Central cone compactification

Central cone compactification

- We consider the space of integral valued quadratic forms:

$$I_n = \{A \in S^n \text{ s.t. } A[x] \in \mathbb{Z} \text{ for all } x \in \mathbb{Z}^n\}$$

  All the forms in I_n have integral coefficients on the diagonal and half integral outside of it.
- The centrally perfect forms are the elements of I_n that are vertices of conv I_n.
- For $A \in I_n$ we have $A[x] \geq 1$. So, $I_n \subset R_n$
- Any root lattice is a vertex both of R_n and conv I_n.
- The centrally perfect forms are known for $n \leq 6$:

dim.	Centrally perfect forms
2	A_2 (Igusa)
3	A_3 (Igusa)
4	A_4, D_4 (Igusa)
5	A_5, D_5 (Namikawa)
6	A_6, D_6, E_6 (Dutour Sikirić)

- By taking the dual we get tessellations of $S^n_{rat,\geq 0}$.

Enumeration of centrally perfect forms

- Suppose that we have a conjecturally correct list of centrally perfect forms $A_1, \ldots, A_m$. Suppose further that for each form A_i we have a conjectural list of neighbors $N(A_i)$.
- We form the cone

$$C(A_i) = \{X - A_i \text{ for } X \in N(A_i)\}$$

  and we compute the orbits of facets of $C(A_i)$.
- For each orbit of facet of representative f we form the corresponding linear form f and solve the Integer Linear Problem

$$f_{opt} = \min_{X \in I_n} f(X)$$

  We have to use GLPK program for that. It is done iteratively since I_n is defined by an infinity of inequalities.
- If $f_{opt} = f(A_i)$ always then the list is correct. If not then the X realizing $f(X) < f(A_i)$ need to be added to the full list.

VII. Voronoi II theory

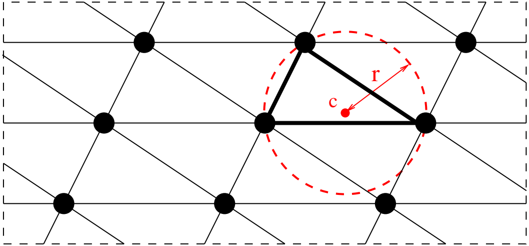
Empty sphere and Delaunay polytopes

A sphere $S(c, r)$ of radius r and center c in an n-dimensional lattice L is said to be an empty sphere if:

(i) $\|v - c\| \geq r$ for all $v \in L$,

(ii) the set $S(c, r) \cap L$ contains $n + 1$ affinely independent points.

A Delaunay polytope P in a lattice L is a polytope, whose vertex-set is $L \cap S(c, r)$.



Equalities and inequalities

- Take $M = G_v$ with $v = (v_1, \dots, v_n)$ a basis of lattice L.
- If $V = (w_1, \dots, w_N)$ with $w_i \in \mathbb{Z}^n$ are the vertices of a Delaunay polytope of empty sphere $S(c, r)$ then:

$$||w_i - c|| = r \ \text{ i.e. } \ w_i^T M w_i - 2 w_i^T M c + c^T M c = r^2$$

- Subtracting one obtains

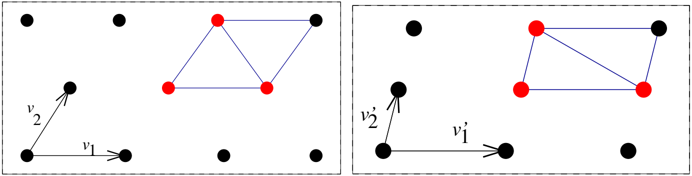
$$\{w_i^T M w_i - w_j^T M w_j\} - 2\{w_i^T - w_j^T\} M c = 0$$

- Inverting matrices, one obtains $Mc = \psi(M)$ with ψ linear and so one gets linear equalities on M.
- Similarly $||w - c|| \geq r$ translates into linear inequalities on M: Take $V = (v_0, \dots, v_n)$ a simplex ($v_i \in \mathbb{Z}^n$), $w \in \mathbb{Z}^n$. If one writes $w = \sum_{i=0}^n \lambda_i v_i$ with $1 = \sum_{i=0}^n \lambda_i$, then one has

$$||w - c|| \geq r \Leftrightarrow w^T M w - \sum_{i=0}^n \lambda_i v_i^T M v_i \geq 0$$

Iso-Delaunay domains

- Take a lattice L and select a basis $v_1, \dots, v_n$.
- We want to assign the Delaunay polytopes of a lattice. Geometrically, this means that



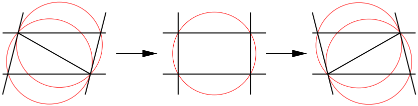
  are part of the same iso-Delaunay domain.
- An iso-Delaunay domain is the assignment of Delaunay polytopes. It is a polyhedral domain of $S^n_{rat,\geq 0}$.

Primitive iso-Delaunay

- If one takes a generic matrix M in $S^n_{>0}$, then all its Delaunay are simplices and so no linear equality are implied on M.
- Hence the corresponding iso-Delaunay domain is of dimension $\frac{n(n+1)}{2}$, they are called primitive

Equivalence and enumeration

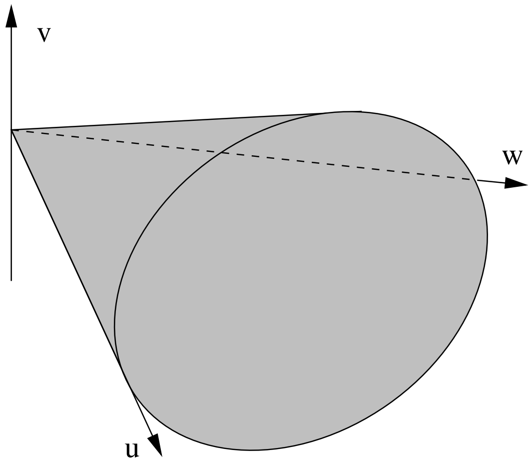
- The group $\mathrm{GL}_n(\mathbb{Z})$ acts on $S^n_{>0}$ by arithmetic equivalence and preserve the primitive iso-Delaunay domains.
- Voronoi proved that after this action, there is a finite number of primitive iso-Delaunay domains.
- Bistellar flipping creates one iso-Delaunay from a given iso-Delaunay domain and a facet of the domain. In dim. 2:



- Enumerating primitive iso-Delaunay domains is done classically:
  - Find one primitive iso-Delaunay domain.
  - Find the adjacent ones and reduce by arithmetic equivalence.
- This is very similar to the Voronoi algorithm for perfect forms.

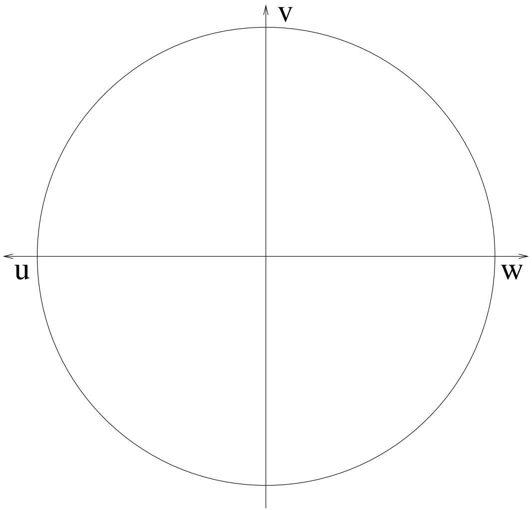
The partition of $S^2_{rat,\geq 0} \subset \mathbb{R}^3$ I

If $q(x,y) = \mathrm{u}x^2 + 2\mathrm{v}xy + \mathrm{w}y^2$ then $q \in S^2_{>0}$ if and only if $\mathrm{v}^2 < \mathrm{uw}$ and $\mathrm{u} > 0$.



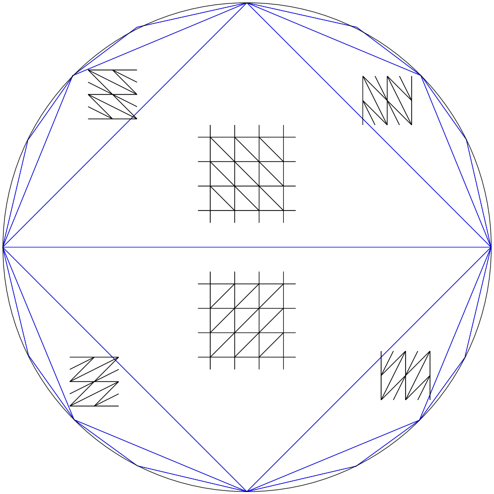
The partition of $S^2_{rat,\geq 0} \subset \mathbb{R}^3$ II

We cut by the plane $\mathrm{u} + \mathrm{w} = 1$ and get a circle representation.



The partition of $S^2_{rat,\geq 0} \subset \mathbb{R}^3$ III

Primitive iso-Delaunay domains in $S^2_{rat,\geq 0}$:



Enumeration of iso-Delaunay domains

- The covering density is equal to the maximum of the circumradius of the Delaunay polytopes.
- In principle if one knows all primitive iso-Delaunay then one can find the best covering lattice.
- A lattice is rigid (Grishukhin & Baranovski) if it is determined by its Delaunay polytopes (iso-Delaunay domain of dimension 1)

dim.	Best covering	Nr. of primitive iso-Delaunay	Nr. of rigid lattices
2	A_2 (Kershner)	1 (Voronoi)	0
3	A_3^* (Bambah)	1 (Voronoi)	0
4	A_4^* (Delone & Ryshkov)	3 (Voronoi)	1
5	A_5^* (Ryshkov & Baranovski)	222 (Engel)	7
6	L_6 (conj. Vallentin)?	$\geq 2.10^8$ (Engel)	≥ 20000

- See for more details
  - A. Schürmann, *Computational geometry of positive definite quadratic forms*, University Lecture Notes, AMS.

THANK

YOU